Alan Tupaj Vista Murrieta High School Website: www.vmhs.net (Click on "Teachers" then "Alan Tupaj")	Differential Equations and Optimization in FRQ's AP Readiness Session 8 - March Answers to examples posted on my website

Examples Solutions

1.

$R(t) = 0$ when $t = 0$, $t = 2.5\pi$, or $t = 7.5\pi$
$R(t) > 0$ on $0 < t < 2.5\pi$
$R(t) < 0$ on $2.5\pi < t < 7.5\pi$
$R(t) > 0$ on $7.5\pi < t < 31$

The absolute maximum number of mosquitoes occurs at $t = 2.5\pi$ or at $t = 31$.

$1000 + \int_0^{2.5\pi} R(t)\,dt = 1039.357,$

There are 964 mosquitoes at $t = 31$, so the maximum number of mosquitoes is 1039, to the nearest whole number.

t = 7.854, t = 23.562
At t = 23.562, total amount = 842.4

2.

$P'(t) = 0$ when $t = 9.183503$ and $t = 10.816497$.

t	$P(t)$
8	0
9.183503	5.088662
10.816497	2.911338
12	8

Entries are being processed most quickly at time $t = 12$.

3.

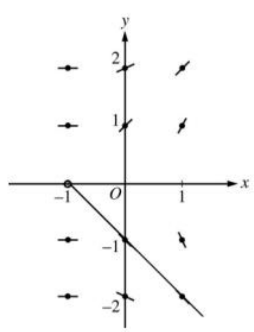
Velocity is 0 at $t = 0$, $t = 9$, and $t = 15$.

t	position at time t
0	0
9	$\frac{9+5}{2} \cdot 20 = 140$
15	$140 - \frac{6+4}{2} \cdot 10 = 90$
18	$90 + \frac{3+2}{2} \cdot 10 = 115$

The squirrel is farthest from building A at time $t = 9$; its greatest distance from the building is 140.

4.

(a)



$3: \begin{cases} 1: \text{zero slopes} \\ 1: \text{nonzero slopes} \\ 1: \text{solution curve through } (0, -1) \end{cases}$

(b) $-1 = \frac{x+1}{y} \Rightarrow y = -x - 1$

$\frac{dy}{dx} = -1$ for all (x, y) with $y = -x - 1$ and $y \neq 0$

1 : description

(c) $\int y\,dy = \int (x+1)\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + x + C$

$\frac{(-2)^2}{2} = \frac{0^2}{2} + 0 + C \Rightarrow C = 2$

$y^2 = x^2 + 2x + 4$

Since the solution goes through $(0, -2)$, y must be negative. Therefore $y = -\sqrt{x^2 + 2x + 4}$.

$5: \begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration
Note: $0/5$ if no separation of variables

5.

(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$.

The answer in part (a) is an underestimate.

$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\,dW = \int \frac{1}{25}\,dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \leq t \leq 20$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Team Practice Solutions

1.

$x'(t) = -e^{-t}\sin t + e^{-t}\cos t = e^{-t}(\cos t - \sin t)$

$x'(t) = 0$ when $\cos t = \sin t$. Therefore, $x'(t) = 0$ on $0 \leq t \leq 2\pi$ for $t = \frac{\pi}{4}$ and $t = \frac{5\pi}{4}$.

The candidates for the absolute minimum are at $t = 0, \frac{\pi}{4}, \frac{5\pi}{4}$, and 2π.

t	$x(t)$
0	$e^0\sin(0) = 0$
$\frac{\pi}{4}$	$e^{-\frac{\pi}{4}}\sin\left(\frac{\pi}{4}\right) > 0$
$\frac{5\pi}{4}$	$e^{-\frac{5\pi}{4}}\sin\left(\frac{5\pi}{4}\right) < 0$
2π	$e^{-2\pi}\sin(2\pi) = 0$

The particle is farthest to the left when $t = \frac{5\pi}{4}$.

2.

$R'(t) = 0$ when $t = 0$ and $t = 1.36296$

The maximum rate may occur at 0, $a = 1.36296$, or 2.

$R(0) = 0$

$R(a) = 854.527$

$R(2) = 120$

The maximum rate occurs when $t = 1.362$ or 1.363.

3.

$Y'(t) = 0$ when $S(t) - R(t) = 0$.
The only value in $[0, 6]$ to satisfy $S(t) = R(t)$
is $a = 5.117865$.

t	$Y(t)$
0	2500
a	2492.3694
6	2493.2766

The amount of sand is a minimum when $t = 5.117$ or
5.118 hours. The minimum value is 2492.369 cubic yards.

4.

The only sign change of g' from positive to negative in the interval is at $x = 2$.

$$g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$$

$$g(2) = 5$$

$$g(7) = 5 + \int_2^{7} g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$$

The maximum value of g for $-3 \le x \le 7$ is $\frac{15}{2}$.

5.

(a) Slope $= \dfrac{(3)(4)}{-8} = -\dfrac{3}{2}$

An equation for the tangent line is $y = -\frac{3}{2}(x-2) - 8$.

$f(1.8) \approx -\frac{3}{2}(1.8-2) - 8 = -7.7$

(b) $\int y\,dy = \int 3x^2\,dx$

$\frac{1}{2}y^2 = x^3 + C$

$\frac{1}{2}(-8)^2 = 2^3 + C \Rightarrow C = 24$

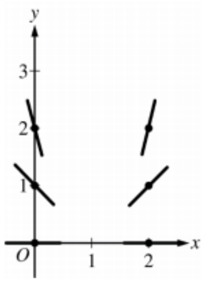
$y^2 = 2\left(x^3 + 24\right) = 2x^3 + 48$

$y = -\sqrt{2x^3 + 48}$

Note: This solution is valid for $x > -\sqrt[3]{24}$.

6.

(a)



$2: \begin{cases} 1: \text{zero slopes} \\ 1: \text{nonzero slopes} \end{cases}$

(b) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,3)} = \frac{3^2}{2-1} = 9$

An equation for the tangent line is $y = 9(x-2)+3$.

$f(2.1) \approx 9(2.1-2)+3 = 3.9$

$2: \begin{cases} 1: \text{tangent line equation} \\ 1: \text{approximation} \end{cases}$

(c) $\frac{1}{y^2}\,dy = \frac{1}{x-1}\,dx$

$\int \frac{1}{y^2}\,dy = \int \frac{1}{x-1}\,dx$

$-\frac{1}{y} = \ln|x-1| + C$

$-\frac{1}{3} = \ln|2-1| + C \Rightarrow C = -\frac{1}{3}$

$-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$

$y = \dfrac{1}{\frac{1}{3} - \ln(x-1)}$

Note: This solution is valid for $1 < x < 1 + e^{1/3}$.

$5: \begin{cases} 1: \text{separation of variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1: \text{solves for } y \end{cases}$

Note: max $3/5$ [1-2-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

7.

(a) $f'(1) = \left.\frac{dy}{dx}\right|_{(1,\,2)} = 8$

An equation of the tangent line is $y = 2 + 8(x-1)$.

$2: \begin{cases} 1: f'(1) \\ 1: \text{answer} \end{cases}$

(b) $f(1.1) \approx 2.8$

Since $y = f(x) > 0$ on the interval $1 \le x < 1.1$,

$\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right) > 0$ on this interval.

Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.

$2: \begin{cases} 1: \text{approximation} \\ 1: \text{conclusion with explanation} \end{cases}$

(c) $\frac{dy}{dx} = xy^3$

$\int \frac{1}{y^3}\,dy = \int x\,dx$

$-\frac{1}{2y^2} = \frac{x^2}{2} + C$

$-\frac{1}{2\cdot 2^2} = \frac{1^2}{2} + C \Rightarrow C = -\frac{5}{8}$

$y^2 = \dfrac{1}{\frac{5}{4} - x^2}$

$f(x) = \dfrac{2}{\sqrt{5-4x^2}}, \quad \frac{-\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$

$5: \begin{cases} 1: \text{separation of variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables